Invading interfaces and blocking surfaces in high-dimensional disordered systems

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We study the high-dimensional properties of an invading front in a disordered medium with random pinning forces. We concentrate on interfaces described by bounded slope models belonging to the quenched Kardar-Parisi-Zhang [Phys. Rev. Lett. **56**, 889 (1986)] universality class. We find a number of qualitative transitions in the behavior of the invasion process as dimensionality increases. In low dimensions $d<6$ the system is characterized by two different roughness exponents: the roughness of individual avalanches and the overall interface roughness. We use the similarity of the dynamics of an avalanche with the dynamics of invasion percolation to show that above $d=6$ avalanches become flat and the invasion is well described as an annealed process with correlated noise. In fact, for $d\geqslant 5$ the overall roughness is the same as the annealed roughness. In very large dimensions, strong fluctuations begin to dominate the size distribution of avalanches; this phenomenon is studied on the Cayley tree, which serves as an infinite dimensional limit. We present numerical simulations in which we measured the values of the critical exponents of the depinning transition, both in finite-dimensional lattices with $d\leqslant 6$ and on the Cayley tree, which support our qualitative predictions. We find that the critical exponents in $d=6$ are very close to their values on the Cayley tree and we conjecture on this basis the existence of a further dimension, where mean-field behavior is obtained. [S1063-651X(97)06308-3]

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I. INTRODUCTION

The problem of interface growth and wrinkling in disordered and noisy systems has been the subject of much interest [1]. Special attention has been given lately to interface growth in disordered systems. Examples of such systems are two-fluid displacement flow in porous media [2], invasion of water into paper [3], and magnetic domain movement in spin systems with quenched disorder. The distinction between such disordered systems where the noise is quenched and cases where the motion is induced by random annealed noise has been an important part of this study [1,4]. It has been known for some time that the critical properties of interfaces in the two kinds of systems are different [1,3–5]. As a consequence of the time independence of disorder in a quenched system, an interface invading a quenched medium may be pinned if the driving force is smaller than some threshold value, while in the annealed version it will always grow. The depinning transition is a critical phenomenon, in which the correlation length diverges as one approaches the threshold driving force value. Thus there is usually a marked distinction between the behavior of the quenched noise problem and the annealed versions, and they define different universality classes.

Advancing interfaces are usually characterized by a roughness exponent χ relating the average width of the interface W to the system size L,

$$W=L^{\chi}. \tag{1}$$

In most cases, the interface has also a self-affine structure characterized by the same roughness exponents. A particular difference between annealed and quenched growth is that the roughness exponent χ_q of an interface growing in quenched disorder is generally different from χ_a, the roughness of the annealed counterpart. An important example is bounded slope growth, which may be described by the Kardar-Parisi-Zhang (KPZ) equation [6] for the annealed case [1]. The quenched model [1,5,7,8] has a different roughness.

An interesting aspect of quenched growth processes is the mapping that exists between such problems and the percolation of random surfaces. In fact, a pinned interface describes a blocking surface that traverses from one side of the system to the other. If the interface is depinned, on the other hand, no such blocking surface exists. In the simplest example of 1+1 dimensions, the blocking surface describes a line in a directed percolation cluster [3,5]. In other dimensions such generalized percolation processes are less well understood. Thus we can identify the directed percolation transition with the depinning transition in two dimensions.

We want to emphasize that the analysis presented in this paper relates to a universality class that is realized in many different contexts. It was found in studying other systems that much useful information and insight can be gained by studying the behavior in spaces of varying dimensionality. This is the case also for the interface growth model studied here. Since the model is rather involved, we discover a number of qualitative transitions as the dimensionality increases.

Turning back to the dynamics of interfaces, we note that critical behavior in such interfaces can be observed using two different driving mechanisms. First, one observes critical properties near the threshold force level p_c, required for the interface depinning. Alternatively, one may drive the system at the point of weakest resistance using a constant infinitesimal current. This driving causes the system to self-organize into a critical state. The critical interface in both cases is that of a self-affine interface with infinite-range correlations. Interfaces produced by the two methods are intimately related. We will use both methods in our discussion.

Analyzing interface growth driven by infinitesimal current, we can define a key concept for the understanding of such processes, namely, the associated process (AP) [8,9].

Associated processes are defined as the domains covered by the growth at periods during which the resistance of the medium is below a certain threshold value. In annealed disorder one cannot define such processes and this is the main source for the distinction between annealed and quenched growth models. We analyze the statistical properties of the APs and in particular their accompanying roughness exponent χ_c, which is defined by the scaling of the height of the APs with respect to their lateral dimension. We stress that χ_c is an independent exponent and may be different from χ_q since the latter is created by a process of stacking independent critical APs. In fact, χ_c is only a lower bound for the overall roughness χ_q [8]. Thus, in general, blocking surfaces have two independent roughness exponents. The first is the global one and the second is the scaling exponents of the voids between the surfaces.

In a previous paper [10] we extended the discussion and analysis of interface growth as a series of APs to an arbitrary number of space dimensions and used the AP concept to derive a scaling theory of interface growth in all dimensions. We argued from general principles that the APs become less rough as the dimensionality increases and eventually become flat objects at a finite dimension d_c. For space dimensions $d>d_c$ the APs are fractal. Note that this dimension is critical in the sense that the scaling relation change qualitatively, reflecting a fundamental change in the dynamical process. However, a second critical dimension can exist where the critical behavior will be the same as in the infinite-dimensional limit.

In this paper we augment the discussion in [10] in several ways. We consider the dynamical process that occurs during the evolution of a critical AP. The set of sites that are active at a certain instant is sparse. Therefore, we conjecture that it is a percolative dynamical process. Using this claim, we can put limits on the maximal possible AP roughness χ_c as a function of d and we are able to show that $d_c=6$ for the quenched KPZ universality class. This is related to the observation in [12] that the dynamical exponent z equals 2 for $d\geqslant 6$.

We next present simulations of the model on lattices of up to six dimensions, verifying that the d_c is indeed equal to 6. We analyze the overall interface growth and show that the interface roughness exponent χ_q is the same as the exponent χ_a of the annealed KPZ equation above $d=4$.

The infinite-dimensional limit of this problem can be analyzed using a realization of the model on a Cayley tree with an additional height coordinate. Section III is devoted to this analysis. We performed simulations on this model and found two main outstanding features. The first one is the existence of strong anomalous behavior, resulting in power-law distribution functions with changing exponents in the subcritical range. Second, near the depinning transition a critical behavior emerges with a strong anomalous background. The values of the scaling exponents seem to be the proper infinite-dimensional limit of the model. In fact, the scaling exponents in six dimensions, which is the highest-dimensional system we analyzed numerically, are quite close to the results on the Cayley tree. This is an indication that the critical dimension of the problem, where all exponents reach their infinite-dimensional limit, is close to 6.

In Sec. III we discuss two variants of the model on the Cayley tree. The first is a straightforward generalization of the bounded slope model we just described, in which the bounded slope condition permits avalanches propagating in all directions, allowing for self-interaction by a reactivation of a site by a daughter site. This self-interaction prevents us from obtaining a closed-form solution. Hence we define a simplified version in which the bounded slope condition is enforced only in a preferred direction. This directed version is simplified enough that analytic calculation can be done, using a mapping to an infinite state branching process. However, the directed version is not a satisfying high-dimensional limit since the anomalous fluctuations mentioned above swamp the critical behavior of the transition and do not allow for an explicit calculation of scaling exponents.

II. THE INVASION PROCESS AND INTERFACE STRUCTURE IN HIGH DIMENSIONS

A. Basic definitions and scaling relations

The discrete Buldyrev-Sneppen model [3,5,7], which belongs to the quenched KPZ universality class, was originally defined on a $(1+1)$-dimensional lattice. At each site there is a quenched random pinning force $f(s)$ distributed uniformly between 0 and 1. There are two related ways to define the invasion process. In the constant current version one has to locate the site of minimal f value on the interface. This site is activated by increasing the interface height at that point. Next one checks whether the activation of this minimal site creates a local slope that is larger than a threshold value. If such an event occurs, the neighboring site is also activated to reinstate a subthreshold local slope. These two operations are carried out repeatedly, creating an advancing interface. Another algorithm for advancing the interface is the activation of all sites on the interface whose value of pinning force f is smaller than a fixed driving force value f_0, followed by avalanches to enforce the bounded slope condition and repeating these operation. If f_0 is chosen such that it is smaller than a critical value p_c, the interface will be pinned eventually; and if $f_0>p_c$, the interface will continue to propagate indefinitely.

A very fruitful way of describing the invasion process by constant current is to introduce the concept of associated processes [8,9]. They are defined by examining the value of f for each activated site. An f_0 AP is a set of activations with f values below the threshold value f_0. An activation above f_0 concludes an f_0 AP and starts a new one. The APs of f_2 may comprise several f_1 APs if $f_1<f_2$. The largest APs are the p_c APs. In the constant current algorithm, no site with $f>p_c$ can be activated. An alternative way to construct an AP is to switch to the constant force f_0 algorithm immediately after updating a site with $f>f_0$. The process will be blocked and the cluster that is created is precisely an AP; this is what relates the two driving mechanisms.

The definitions of the model and the two driving methods can be carried through unchanged to lattices of higher dimensionality. One defines an interface height coordinate in each site and performs the invasion process according to the rules just described. The lattice does not have to be an Euclidean lattice, and we will consider interfaces growing on a Cayley tree in Sec. III.

For f_0 close to p_c the APs display critical behavior, where $\Delta f \equiv p_c - f_0$ measures the closeness to criticality of the AP. We are interested in the distribution of AP sizes (or masses) s, defined as the number of activations in the AP, and in the rms lateral dimension $r_\parallel$. The following scaling forms are known to be valid for d-dimensional APs near f_c:

$$p(s) = s^{-\tau} g(s/\Delta f^{-\nu}), \quad p(r_\parallel) = r_\parallel^{-\tau_\parallel} g_\parallel(r_\parallel/\Delta f^{-\nu_\parallel}), \quad (2)$$

and the fractal dimension D of the APs is defined by

$$s \sim r_\parallel^D. \quad (3)$$

The scaling relations

$$\nu_\parallel = \nu/D, \quad \tau_\parallel - 1 = D(\tau - 1) \quad (4)$$

follow immediately from the definitions (2) and (3). If $D > d$, the APs are rough objects, that is, they have a width $r_\perp$ related to $r_\parallel$ by

$$r_\perp \sim r_\parallel^{\chi_c}, \quad (5)$$

where $\chi_c = D - d$.

An exponent relation proved in [9] using the properties of the model is

$$1 = \nu_\parallel(d + 1 - \tau_\parallel). \quad (6)$$

This relation reduces the number of independent static exponents to 2 and remains valid as long as the AP roughness exponent χ_c is positive. A relation equivalent to Eq. (6) is

$$\gamma = 1 + \nu\chi_c, \quad (7)$$

where $\gamma = \nu(2 - \tau)$ is the scaling exponent of the average cluster size with respect to Δf.

We will show below that there exists a finite dimension d_c such that $\chi_c = 0$ when $d > d_c$. Above d_c the scaling relation (7) becomes simply $\gamma = 1$. However, the other exponents and the fractal dimension continue to depend on the dimension and therefore d_c does not mark the transition to mean-field behavior.

The geometric structure of an AP is qualitatively different in $d > d_c$. Studying the rules of the model indicates that whenever χ_c is positive, the APs cannot be fractal since rough clusters have a typical width $r_\perp$, which through avalanches prevents the formation of holes with diameter smaller than $r_\perp$. For $d \leqslant d_c$ the APs may be multiply connected, but not fractal. Above d_c, since $D \leqslant d$, the AP become objects with order 1 thickness, which may fractalize by having holes of any size.

B. Three types of roughness exponents

In the preceding subsection we defined the roughness exponent χ_c that characterizes the statistics of the APs. It is important to realize that χ_c is not necessarily equal to the exponent χ_q defined in Eq. (1), which describes the interface as a whole. However, a simple consideration shows that χ_q cannot be smaller than the value of χ_c. In fact, from time to time an AP is generated that encompasses the whole system and then the entire interface identifies with a critical AP (see also [8]). The interface growth is also related to the annealed process since the stacking of APs is similar in spirit to the process described by annealed equations. This issue is dealt with in more detail below, but here we would like to note that it follows from our discussion that the roughness of the annealed dynamics χ_a is also a lower bound for χ_q, so we may write

$$\chi_q \geqslant \max(\chi_a, \chi_c). \quad (8)$$

Since the quenched growth includes a nontrivial interaction between the processes defining χ_a and χ_c its roughness can actually be larger than both χ's.

As an example one can consider the $(1+1)$-dimensional case. Since the APs are related to directed percolation, their roughness is $\chi_c = 0.63$ [3,5]. The roughness for the annealed dynamics is $\chi_a = 0.5$, so that χ_c dominates χ_a. In fact, it has been found (see [8]) that in this case the system roughness χ_q is slightly larger than χ_c. In higher-dimensional systems χ_c decreases and the difference between χ_q and χ_c becomes more pronounced.

As an application of this distinction, we note that when constant force driving is used, starting from a *flat* interface as the initial condition, the created clusters will have roughness χ_q rather than χ_c. In fact, the clusters will consist of many APs since the initial surface cuts through them. This difference will be very manifest in high dimensions, where the APs are flat and fractal, whereas the constant force clusters are rough [11].

C. Dependence of χ_c on dimensionality

It is apparent that growth in APs can happen in two ways: either upward by activations or sideways via avalanches. The sideways action becomes more important as the dimensionality increases and χ_c therefore decreases as a function of d. A naive expectation based on this scenario is that in sufficiently high dimensions the sideways growth becomes non-self-interacting. If this argument were true, the APs should belong to the percolation universality class in large enough dimensions. This simple behavior does not occur since the bounded slope condition induces a self-interaction that does not decrease when d increases, as we explain in detail in Sec. III, where we analyze the model on the Cayley tree.

There is, however, a subtle connection between the dynamics of a growing AP and the dynamics of a percolation cluster. In the following subsection we exploit this analogy to argue that in some finite dimension d_c the APs become flat objects with $\chi_c = 0$. As explained above, d_c does not mark the transition to mean-field behavior. Some exponents, such as the AP fractal dimension D, change as a function of d above d_c. Hence, for $d > d_c$ the APs are fractal objects with $D < d$ (see Sec. II A). We expect that D saturates at some finite value D_∞, as indicated by the Cayley tree analysis presented in Sec. III.

D. Dynamical scaling and critical dimension

The discussion presented so far has concentrated on the AP static exponents such as τ and D. It is interesting that

TABLE I. Results of numerical simulation of bounded slope models. The finite-d entries refer to simulations on hypercubic lattices and the $d=\infty$ entry refers to simulations on a Cayley tree with coordination number 3. The exponents D, τ were directly measured, as well as γ in the finite-dimensional case. The value of τ was calculated from D and $\tau_\parallel$ using Eq. (4) in the finite-dimensional simulations and measured directly in the Cayley tree simulation. The value of χ (the overall roughness) was not measured, but is given for reference. The numbers in parentheses are errors in the last digit. We have checked that the numerical values of exponents presented here satisfy all the theoretical scaling relations that connect them.

d	p_c	γ	D	$\tau_\parallel$	τ	χ [8,11]
1	0.4610(2)	2.03(3)	1.64(2)	1.42(2)	1.25	0.67
2	0.2002(2)	1.53(2)	2.52(2)	2.11(2)	1.44	0.50
3	0.1148(3)	1.29(2)	3.40(2)	2.86(2)	1.55	0.38
4	0.0785(5)	1.21(1)	4.24(3)	3.58(2)	1.61	0.27
5	0.0583(2)	1.13(2)	5.07(3)	4.22(3)	1.64	0.25
6	0.0445(5)	1.08(2)	6.00(5)	5.05(5)	1.68	0.2
∞	0.150(1)	1	7(1)	6.6(2)	1.75(5)	0

considerations involving dynamic exponents provide further information on static exponents as well. Although the arguments given here are not rigorous, we believe that they still give a consistent and informative picture of the invasion process.

The dynamical exponent z is defined as follows. Consider an AP initiated at some site following its growth activating at each time step all the available sites at that moment. The lateral size of the AP grows as a power law in time

$$t \sim r^z. \tag{9}$$

It is easy to show that the fractal dimension of the set of activated sites at any instant is

$$D_{ac} = d + \chi_c - z. \tag{10}$$

Consider the ongoing process of activation on the surface of the cluster. Since the fractal dimension D_{ac} is smaller than d (since $\chi_c < 1 < z$), the set of active sites is so sparse that the time interval between subsequent activations of a single site will be very long (in fact, we can show that it scales as $r^{z\chi_c/(z-1)}$). This indicates that the dynamical process of the AP growth is a percolative process, similar to the process the dynamical invasion into a percolation cluster. The first conclusion from this argument is that $z = z_{per}$, the dynamical exponent of invasion percolation. This relation has been suggested in [12] and verified numerically in dimensions $d \leqslant 6$.

The AP dynamics in fact resembles a set of invasion percolation processes occurring simultaneously, so that D_{ac} may be larger than the dimension of the set of active sites in a percolation process $D_{per} - z_{per}$. However, D_{ac} cannot be larger than the fractal dimension of a full percolation cluster D_{per} without completely destroying the dynamical picture. Hence consistency demands that

$$D_{ac} = d + \chi_c - z \leqslant D_{per} \tag{11}$$

and thus

$$\chi_c \leqslant D_{per} + z_{per} - d \leqslant 6 - d. \tag{12}$$

In the last inequality we used the largest possible values for D_{per} and z_{per}, which are the mean-field values. The inequality (12) gives an upper bound for the AP roughness χ_c in each dimension. We verified that this bound indeed holds for the numerical values of χ_c and is actually rather sharp (see Table I in Sec. II E). Moreover, Eq. (12) implies that when $d \geqslant 6$, χ_c must be 0, so we have

$$d_c \leqslant 6. \tag{13}$$

We note that for $d > d_c$ our arguments become irrelevant since the AP is then a fractal object and its growth in this case will occur mostly in the boundary and not in its bulk. Therefore, we expect that the process is no longer a percolative one. It is amusing to note that for this model only the low-dimensional processes in $d \leqslant d_c = 6$ are percolative.

E. Numerical simulations in high dimensions

We performed numerical simulations of the Buldyrev-Sneppen model and compared them with the previous theoretical scenario. The simulations were of either of two kinds: interface growth by constant current on a hypercubic lattice with periodic boundary conditions and invasion of a single cluster in a Cayley tree as an effective infinite-dimensional lattice.

In each of the finite-dimensional simulations the number of activations was of the order of 10^{10} and thus statistical errors were rather small. However, in high-dimensional systems we were severely constrained by memory considerations. For example, in six dimensions, the highest dimensionality in our simulations, the hypercube measured only 16 sites in each directions. Thus finite-size effects were very strong in the high-dimensional cases, leaving a small scaling range still observable. The numerical investigations on the Cayley tree are described in detail in Sec. III.

We measured directly the exponents $\tau_\parallel$, γ, and D. These exponents are presented in Table I. Since only two exponents are independent in each case, measuring three different exponents provides a consistency check. The values displayed in Table I obey the scaling relations (4) and (6), within numerical error.

The most remarkable feature in Table I is the monotonic decrease of $\chi_c = D - d$ as a function of d. In six dimensions the APs become flat to within numerical accuracy, indicating that $d_c \sim 6$. This is in accord with our previous estimate for the value of d_c based on dynamic scaling. Since it has been also confirmed in [12] that $z=z_{per}$, we believe that our conjecture about the nature of the dynamical process of invasion is very plausible. We also note that in four dimensions the measured χ_c becomes smaller than χ_a, the annealed roughness. This implies [cf. Eq. (8)] that the interface roughness χ is dominated by an annealed mechanism. In the next subsection we discuss this point and analyze in detail its consequences. Another feature evident from Table I is that in six dimensions the scaling exponents seem well converged toward their infinite-dimensional values.

F. Overall interface roughness

We now turn our attention to the scaling of the growing interface as a whole. Above d_c the APs are flat and the invasion process becomes very similar to the annealed process: Each AP activates each member of a set of sites once, or at most a few times. In this regime the interface may be described by the annealed KPZ equation [1,6]

$$\partial_t h(\mathbf{x},t)=\nabla^2 h(\mathbf{x},t)+\lambda[\nabla h(\mathbf{x},t)]^2+\eta(\mathbf{x},t), \tag{14}$$

where h is the height and the random noise η describes the activation of sites by the flat APs. Since the activation of sites by APs favors nearby sites, η has spatial correlations derived from the distribution and structure of the flat APs. We expect that in large enough dimensions, the spatial correlations of η will become irrelevant and that the critical exponents of the interface will identify with those of KPZ dynamics with δ-correlated noise. We thus define *interface critical dimension* d_i, such that $\chi_a=\chi_q$ for $d\geqslant d_i$. When observing the invasion process of the interface as a whole, rather than the APs, d_i signifies the dimensionality where the effects of noise quenching become irrelevant.

It is not difficult to estimate the statistics of the correlated noise η, when $d>d_c$, in terms of the AP properties. Since the APs are not correlated (for $f_0=f_c$, see [9]), η is Gaussian and its value at two points is correlated only if both sites were activated in the same AP. Therefore, the correlation function may be estimated by the probability of such an event,

$$\langle\eta(\mathbf{x})\eta(\mathbf{y})\rangle\sim P(r_\parallel\geqslant|\mathbf{y}-\mathbf{x}|)|\mathbf{y}-\mathbf{x}|^{D-d}\sim|\mathbf{y}-\mathbf{x}|^{D(2-\tau)-d}. \tag{15}$$

We now analyze the effects of those correlations, using a k-space representation of the problem. We make the following definitions. Let

$$h(\mathbf{k},\omega)=\int d^d\mathbf{x}\; dt\, h(\mathbf{x},t)e^{i\mathbf{k}\cdot\mathbf{x}+i\omega t}, \tag{16}$$

with $\eta(\mathbf{k},\omega)$ defined similarly. The response function is

$$G(\mathbf{k},\omega)\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')=\langle\delta h(\mathbf{k},\omega)/\delta\eta(\mathbf{k}',\omega')\rangle \tag{17}$$

and we assume that it has the scaling form

$$G(\mathbf{k},\omega)=k^{-z}g(\omega/k^z), \tag{18}$$

where z is the dynamical exponent of annealed growth. The noise correlation in k space is

$$\langle\eta(\mathbf{k},\omega)\eta(\mathbf{k}',\omega')\rangle=k^{\alpha}\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega'), \tag{19}$$

where $\alpha=D(\tau-2)$.

The roughness of an interface described by Eq. (14) is generated by a combination of the noise correlations and the intrinsic roughness stemming from the nonlinear process. It is possible to give a naive calculation for the correlation-induced roughness that is a lower bound to the actual roughness. To this end we define the ‘‘bare’’ height field as

$$h_0(\mathbf{k},\omega)=G(\mathbf{k},\omega)\eta(\mathbf{k},\omega). \tag{20}$$

The bare correlation function is

$$\begin{aligned}\langle h_0(\mathbf{k},\omega)h_0(\mathbf{k}',\omega')\rangle&=G(\mathbf{k},\omega)G(\mathbf{k}',\omega')\\&\quad\times\langle\eta(\mathbf{k},\omega)\eta(\mathbf{k}',\omega')\rangle\\&\sim k^{-2z+\alpha}g(\omega/k^z)^2\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')\end{aligned} \tag{21}$$

and the simultaneous bare correlation is given by

$$\langle h_0(\mathbf{k},t)h_0(\mathbf{k}',t)\rangle=k^{-2z+\alpha}\int d\omega\; g(\omega/k^z)^2\sim k^{-z+\alpha}, \tag{22}$$

so we get the bare roughness

$$2\chi_0=\max(z-\alpha-d,0). \tag{23}$$

Thus the correlation-induced roughness dominates the intrinsic roughness unless

$$2\chi>z-\alpha-d. \tag{24}$$

One can check that the condition for irrelevance (24) is always satisfied for uncorrelated noise $(\alpha=0)$ and that it also reproduces the known condition for irrelevance of the noise correlation in one dimension, $\alpha>-1/2$ [1].

The condition for irrelevance (24) was derived for $d\geqslant d_c$ and is satisfied in this range (see Table I and [13] for values of χ and z for high-dimensional KPZ dynamics); we can thus conclude that $d_i\leqslant 6$. The measured value of χ_c becomes equal to numerical measurements of χ_a at four dimensions within numerical accuracy, so $d_i\geqslant 4$. Since $\chi_c\leqslant 1$ in five dimensions, it is probably safe to use the criterion (24) also for $d=5$. The criterion is satisfied so that we get the final estimate

$$4\leqslant d_i\leqslant 5. \tag{25}$$

Independent measurements of the overall interface roughness [11] indicate that this is indeed the case; see Table I. We conclude that above four dimensions the roughness exponent of the overall quenched growth becomes identical to the annealed growth exponent.

III. CAYLEY TREE

In many statistical physics problems, the infinite-dimensional limit is obtained by defining the model on the Cayley tree. In problems with an upper critical dimension the Cayley tree solution gives the mean-field exponents.

The recursive structure of the Cayley tree simplifies the problems because of the lack of a self-interaction and in many cases allows one to obtain a closed-form solution. A well-known example is the solution of percolation on the Cayley tree [14]. This solution is instructive since it reveals some of the characteristics of the percolation clusters. We are interested in the analogous problem for interface growth. In this case, however, it will be apparent that Cayley tree realizations of this model might reveal properties that are subdominant for any finite-dimensional realization of the model. We shall show that strong fluctuations may potentially destroy the usual kind of phase transition and may lead to anomalous critical properties. The Cayley tree realization that we define here displays a simple critical behavior with a strong background of anomalous behavior, but in other examples the anomalous behavior dominates.

We define the invasion process on the Cayley tree as follows. The interface is defined by assigning a height to each site of the Cayley tree. As in the finite-dimensional case, there is a quenched random force f that depends on the site in question and the height of the interface. A local slope is defined between every pair of neighboring sites and the bounded slope condition prevents the local slope from increasing beyond a critical value by advancing the sites that violate this condition. As usual, there is a critical value of force f_c, above which there is as finite probability to create an infinite cluster.

The avalanche propagates upward and outward from the original site. Since we expect critical avalanches to be flat, propagation of avalanches will be mainly in the outward direction. However, two additional processes complicate the dynamics. First, the bounded slope rule implies that repeated activation of a single site is followed by activation of all its near neighbors and the number of these neighbors grows exponentially with the number of repeated activations of the original site. Second, since the activations are isotropic, sites may be activated, create subavalanches, and then be reactivated by a subavalanche created by their children. Such backward activations creates self-interaction in the dynamics and is one of the main reasons for the difficulty in analyzing the model in a quantitative way. This self-interaction is the reason that the model on a one-dimensional chain, being a special case of the Cayley tree, is not trivially solvable.

The general origin of the anomalous critical behavior in the size distribution of the clusters on the Cayley was discussed briefly in a previous paper [10]. We will give an example below, where this mechanism can be studied analytically, but first we would like to present the qualitative arguments since they have an essential role in the behavior of the phase transition.

Consider an interface that is initially flat and a growth process initiated from a single site with a fixed value of f_0. There is a finite probability that the same site will be activated once more, activating the q nearest neighbors obeying the bounded rule. h subsequent activations of the same site will result in activation of the $q(q-1)^h$ neighboring sites, again as a result of the bounded slope condition. We observe that the number of activations in this subprocess grows exponentially with h. However, since the probability for each of the activation is f_0, the probability for such an event is f_0^h, which is exponentially small. Thus, even though the probability to create such a height difference decreases exponentially with h, the effect might become important because of the exponential avalanches that follow from such events. This effect can dominate the probability distribution of s.

The actual number of sites that are activated by the original site of height h is larger than the previous estimate since other sites can be activated by the exponential subavalanche. Another complication arises since the actual interface is not necessarily flat. However, since it is statistically flat, a sufficiently long sequence of activations should have an exponential effect. Therefore, we estimate the mean number of activations $\overline{s(h)}$ resulting from the event that a single site is of height h by

$$\overline{s(h)} \sim \mu^h, \tag{26}$$

with $\mu \geqslant q-1$. On the other hand, the probability that a site in an f_0 avalanche reaches a height h should decrease like $\tilde{f}^h$ with $\tilde{f}$ larger than but of the order of f_0. We can combine these two estimates to find the probability of an avalanche of size s,

$$\text{Prob}[\overline{s(h)} \leqslant s \leqslant \overline{s(h+1)}] \sim \tilde{f}^h; \tag{27}$$

it follows that

$$P(s) \sim s^{(\ln \tilde{f}/\ln \mu)-1} \equiv s^{\tau(f_0)}. \tag{28}$$

Thus, on the Cayley tree, the tail of the distribution of the AP sizes is always a power law, with a nonuniversal f_0-dependent exponent.

Near the phase transition there are two possible scenarios. Either the transition large clusters are created by the present mechanism or they are the result of critical fluctuations. In the first case there is no typical scale for the mass of the clusters, as follows from Eq. (28). However, there is a typical *length* scale that remains finite in the transition.

The second case may be realized if for sufficiently small $|f_0 - f_c|$ the probability to create a very large critical cluster is larger than the probability displayed in Eq. (28). Thus there will be a crossover scale, beyond which normal critical behavior dominates. In particular, there will be typical scales for the mass and (chemical) length of the clusters, which diverge at f_c, and the clusters will have a finite fractal dimension. Since the mechanism that creates anomalous clusters is not universal, whether or not normal critical behavior is realized depends on the specific realization.

In the following subsections we are going to discuss, in addition to the model that we have already presented, two additional realizations of the bounded slope model on the Cayley tree. One realization was defined and analyzed in [11]; in it the interface grows into the tree instead of in an additional direction. The other realization, to be defined in detail below, is a directed version of the model defined above, where backward activations are disabled. These two alternative realizations are integrable and display anomalous critical behavior. The original model, however, displays numerical evidence for crossover to standard critical behavior.

A. Numerical analysis on the Cayley tree

We performed Monte Carlo simulations of the bounded slope model on the Cayley tree with coordination number 3.

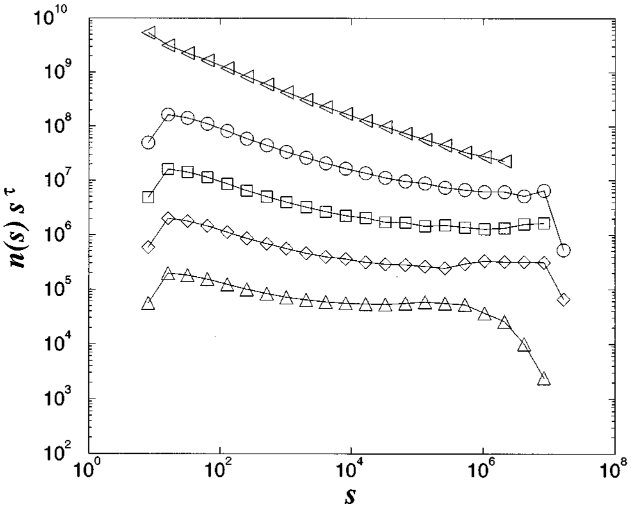

FIG. 1. Distribution of cluster sizes $n(s)$ multiplied by the critical distribution s^{τ_c} (arbitrary units), given as a function of s. We present distributions for forcing levels $f_0=0.141$, 0.145, 0.148, $0.150\sim p_c$, and 0.155, displayed with higher values of f below lower ones. For low values of f, below p_c, one can observe pure anomalous power-law distributions, while as f increases a new feature becomes more and more apparent for large s, as one approaches p_c. The new power law seems to be independent of f near p_c. Above p_c an exponential cutoff appears for large s.

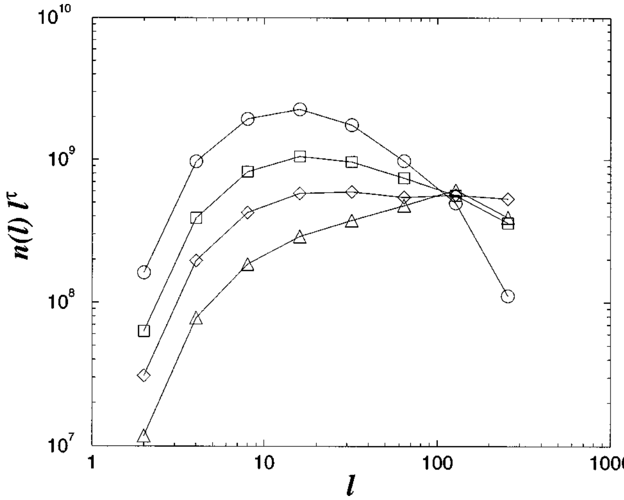

FIG. 2. Distribution of clusters according to their chemical length ℓ, multiplied by the critical distribution ℓ^{τ_ℓ}, for values of f corresponding to those displayed in Fig. 1, except the smallest value. One can observe clearly that the distribution falls faster than a power law for $f<p_c$ and is power-law-like at p_c.

We used the constant driving force algorithm version, i.e., for each site that was a candidate for activation, a random number between 0 and 1 was compared with a constant force f_0 to decide whether the site should be activated. This process was repeated until the cluster either was blocked or exceeded an upper bound in size.

We collected the statistics of clusters in terms of the cluster size s and in terms of the maximal chemical distance from the source site ℓ. In Fig. 1 we give examples of the size distribution for several values of f_0. We observe three qualitative types of behavior, which can be explained in terms of the interplay between critical and anomalous fluctuations described above. In particular, the size distribution very near the critical value of forcing displays a crossover behavior, which we interpret as the emergence of normal critical behavior.

The distribution functions for a subcritical driving force display a clear power-law tail without a cutoff, but with an f_0-dependent exponent $\tau(f_0)$ [see Eq. (28)]. The clusters in the tails of the subcritical distributions were created by the mechanism described in Sec. III. An unusual property that follows from the asymptotic form of the subcritical distributions is the divergence of the average cluster size $\langle s\rangle$ when $\tau<2$. This property has no significance beyond the fact that the size distributions have no cutoff in the subcritical range.

The supercritical size distributions, on the other hand, display well-defined cutoffs. The presence of these cutoffs stems from exactly the same reasons as the supercritical cutoff in usual critical phenomena: Once a cluster reaches a certain size, it becomes very unlikely that it will not become an infinite cluster; this is true regardless of the mechanism that creates the power-law tails in the subcritical range.

Very close to p_c ($f_0=0.150$) a third type of distribution is observed. There is sub-asymptotic power-law tail with an exponent $\tau(f_0)$ characteristic of the subcritical distributions, but at a certain value of s there is a crossover to a different power-law distribution with an exponent $\tau_c=1.75\pm0.05$. We attribute the crossover to the emergence of normal critical behavior, which is masked by the anomalous fluctuations for other values of f_0. The fact that very large clusters near f_c are generated mainly by critical fluctuations signifies that the transition itself is of normal type.

An analysis of the length distributions of clusters $p(\ell)$, displayed in Fig. 2, supports the picture obtained from the analysis of mass distribution. The subcritical distributions decay faster than a power law, but for $f_0\sim f_c$ the distribution decays asymptotically as a power law with a characteristic exponent $\tau_\ell=3.8\pm0.1$, consistent with normal critical behavior at f_c.

We conclude that the depinning transition of the bounded slope model on the Cayley tree is a critical phenomenon with a very strong background of anomalous fluctuations. Using the critical exponents τ_c and τ_ℓ, one may also calculate the fractal dimension of critical the clusters $D=2(\tau_\ell-1)/(\tau_c-1)\sim7$. The large value of D shows that this phase transition is not in the percolation transition universality class. The critical exponents are also listed for reference in Table I.

B. A solvable model

It was remarked above that the main obstacle to obtaining an exact solution for the invasion model on the Cayley tree is self-interaction through reactivation of a site by a ''child'' site. A natural way to overcome this difficulty is to define a model in which there is a preferred direction, ''backward,'' in which the bounded slope rule is not enforced. The mechanism that creates anomalous large fluctuations is present in this simplified directed model and a more precise analysis of this feature is possible. However, in contrast with the ''isotropic'' Cayley tree model that was discussed in the preceding subsection, we show that in the directed model anomalous fluctuations dominate up to the transition and normal critical behavior is not observed. In addition to the directed model, we also discuss briefly the model presented in [11] and reach similar conclusions.

The directed model is therefore defined by precisely the same rules as the original model defined above, except that the slope may be arbitrarily large when the lower point is closer to the root. Thus defined, the process can be carried out also in the following manner. Pick a root site on the tree and let it be activated repeatedly until further activation is blocked by a pinning force larger than some fixed driving force p_0. The probability that a height h will be reached by this process is $p_0^h(1-p_0)$. Since the bounded slope rule works only in the forward direction, the interface height determined by this process may not be changed anymore and is statistically independent of the interface height at any other site. Next, the same process is carried out for each of the child sites of the root, starting from height $h-1$ if $h\geqslant 2$; otherwise the invasion process stops. The process continues analogously for each of the children sites either until all the sites are blocked, or indefinitely, creating an infinite cluster (this can happen only when $p_0 > p_c$).

The present model is equivalent to a branching process [15] and we may use its recursive structure for an analysis, similar to the analysis of Cayley tree percolation. However, the model defines a universality class different from percolation since it is a branching process with an infinite number of states: Each possible value of the interface height corresponds to a different branching process state.

The mass distribution of clusters of the directed model may be calculated as follows. We denote the probability that the total mass of a cluster is s given that the interface height at the root site is h by $p(s|h)$. This function also describes the probability that, given a child site with interface height h, it will generate a subcluster of mass s. Since the total mass of the cluster is equal to the sum of the masses of all the subclusters plus the number of activations at the root, we have

$$p(s|h)=\sum_{s_1+\cdots+s_q+h=s}\ \sum_{h_1,\ldots,h_q} p(s_1|h_1)p(h_1|h)p(s_2|h_2) \times p(h_2|h)\cdots p(s_q|h_q)p(h_q|h). \tag{29}$$

The right-hand-side of Eq. (29) is a sum of products of expressions of the form

$$f(s|h-1)\equiv\sum_{h'} p(s|h')p(h'|h) = \sum_{h'=0}^{\infty} (1-p_0)p_0^{h'} p(s|h'+h-1). \tag{30}$$

It is possible to invert Eq. (30) to express $p(s|h)$ in terms of f and rewrite Eq. (29) as

$$(1-p_0)^{-1}[-p_0 f(s|h+1)+f(s|h)] = \sum_{s_1+\cdots+s_q+h=s} f(s_1|h-1)f(s_2|h-1)\cdots f(s_q|h-1). \tag{31}$$

The Laplace transform of Eq. (31) is

$$(1-p_0)^{-1}[-p_0 g(x|h+1)+g(x|h)] = x^h g(x|h-1)^q, \tag{32}$$

where $g(x|h)\equiv\Sigma_s x^s f(s|h)$ is the Laplace transform of $f(s|h)$. The solution g of this nonlinear difference equation contains all the information about the mass distribution. In particular, many properties of the distribution $p(s|h)$ follow from the behavior of $g(x|h)$ near $x=1$. For example, the probability to create an infinite cluster is

$$p_\infty(h)=1-\frac{g(1|h)-p_0 g(1|h+1)}{1-p_0}. \tag{33}$$

Similarly, the expected cluster size starting at height h is

$$\langle s|h\rangle = \frac{\partial_x g(1|h)-p_0\partial_x g(1|h+1)}{1-p_0}. \tag{34}$$

$\langle s|h\rangle$ obeys a linear difference equation obtained by linearizing Eq. (32) near $g(1|h)$. For small enough p_0, $g(1|h)$ must be 1 and we can linearize Eq. (32) near $x=1$ to get

$$\frac{-p_0\partial_x g(1|h+1)+\partial_x g(1|h)}{(1-p_0)} = q\,\partial_x g(1|h-1)+h, \tag{35}$$

which can be solved in closed form, yielding expressions for $\langle s|h\rangle$. The solution ceases to be realizable when $p_0=p_c$, where p_c is defined by

$$2qp_c(1-p_c)=1. \tag{36}$$

This implies that for $p_0 > p_c$, $g(1|h)\neq 1$, so that p_c indeed is the critical value of p_0.

Further properties may obtained by solving Eq. (32) numerically. One finds that for p_0 near p_c and $1-x$ small

$$g(x|h)\sim 1+(1-x)g_1(h)+g_s(h)(1-x)^\tau, \tag{37}$$

where $2<\tau<3$. This type of asymptotic behavior implies, using the Tauberian theorems [11,16], that $f(s|h)$ and therefore also $p(s|h)$ behave asymptotically as $s^{-\tau}$ for large s and in particular that $\langle s|h\rangle$ is finite but $\langle s^2|h\rangle$ diverges for $p_0\leqslant p_c$ (near p_c). For example, for $q=2$ and $p_0=p_c$ we find that $\tau=2.2\pm 0.05$

We observe that the subcritical mass distribution of the directed model is quite similar to that of the original model, with a p_0-dependent power-law tail, but in contrast with the original model, this behavior persists for values of p_0 up to and including p_c. This indicates that the directed model belongs to the second case outlined at the beginning of this section, where the phase transition is dominated by the anomalous fluctuations caused by exponentially large compact avalanches. The same conclusion also arises from the analysis of the length distribution of the clusters, presented next.

The analysis of the cluster length is more involved than that of the cluster mass. Using the analogy with an infinite state branching process, we interpret the sites at a given chemical distance ℓ from the root as the ℓth generation of the process. The population of the process is defined by the values of the interface height, where each height corresponds to a different species of the branching process. Thus we may define the ℓth generation population vector $\mathbf{Z}_\ell=(Z_\ell^{(1)},Z_\ell^{(2)},\ldots)$, where $Z_\ell^{(1)}$ gives the number of sites at chemical distance ℓ from the root with interface height equal to 1, etc. The branching process is defined by giving the probability $p_n(\mathbf{Z})$ that an individual of type n will generate a set $\mathbf{Z}=(Z^{(1)},Z^{(2)},\ldots)$ of individuals in the next generation. The values of these probabilities follow from the definition of the directed model. Since we were mainly interested in the asymptotics of the length distribution of clusters, we analyzed the dependence of the moments $\langle Z_\ell^{(n)}\rangle$ and $\langle Z_\ell^{(n)}Z_\ell^{(m)}\rangle$ on ℓ. Such moments average both over clusters shorter than ℓ, contributing 0, and over clusters longer than or equal to ℓ. For critical clusters we expect that the distribution of $\mathbf{Z}_\ell$ does not depend on the cluster size, given that cluster is longer or equal to ℓ. Thus we may write

$$\langle Z_\ell^{(n)}\rangle=p(\ell)\langle Z_\ell^{(n)}|\mathbf{Z}_\ell\neq 0\rangle\sim c_n p(\ell)s(\ell), \qquad (38)$$

where $p(\ell)$ gives the probability that a cluster is longer than ℓ and $s(\ell)$ is the average number of sites at a distance ℓ from the root. $s(\ell)$ would be of the form $\ell^{D_\ell-1}$ if the critical clusters have a finite fractal dimension D_ℓ. Similarly,

$$\langle Z_\ell^{(n)}Z_\ell^{(m)}\rangle\sim c_{n,m}p(\ell)s(\ell)^2, \qquad (39)$$

so that we may deduce the behaviors of $p(\ell)$ and $s(\ell)$ from those of the first and second moments of $\mathbf{Z}$.

Explicit calculations, not reproduced here, show that first moments behave as a power law for large ℓ, while second moments grow exponentially. This implies, using Eqs. (38) and (39), that $s(\ell)$ grows exponentially and $p(\ell)$ decays exponentially with ℓ. In other words, the clusters have a characteristic length and have an infinite fractal dimension. These properties are expected for a depinning transition dominated by anomalous fluctuations, as already indicated by the analysis of cluster mass distribution. We conclude that in the directed model critical fluctuations are too weak to be observed even near p_c.

An alternative version for defining the bounded slope model on the Cayley tree was given in [11]. In this version, the Cayley tree is divided into sets of sites with a fixed height and the invasion process occurs directly from one site to its neighbor rather than by raising the interface in a single site as in the version presented in the present paper. Reference [11] presents an analysis of the cluster mass distribution. As in the models presented here, the mass distribution displays nonuniversal power-law asymptotics, stemming from anomalous fluctuations. Since there is no indication of a crossover to a different type of asymptotics near p_c, we conclude that in this model the transition is anomalous.

C. Consequences for large but finite *d*

In many cases one uses information from Cayley tree realizations of statistical models to deduce properties of high-dimensional realizations. In particular, one may hope to find ''mean-field'' behavior, which should be valid above a finite critical dimension d_c, if such exists.

In the case of the bounded slope models, however, we have a dilemma: The behavior on the Cayley tree is complicated by the creation of exponentially large avalanches, so we observe two types of critical behavior. In some cases we had a fluctuation-dominated phase transition and in others the phase transition is of standard critical type.

We believe that the properties observed on the symmetric Cayley tree should serve as a better guide to the finite-dimensional case, for two reasons. First, the symmetric model we used is closer in its definition to high-dimensional bounded slope models than both the directed model we solved and the tree model suggested in [11] since it has a definite height and symmetric activations.

The second reason follows by examining the process that creates the anomalous avalanches in large but finite d. The analogous arguments that led to the conclusion that the cluster mass distribution has power-law asymptotics [cf. Eq. (28)] repeated in the case of finite d yield a distribution that behaves asymptotically as a stretched exponential $\exp(-s^{1/d})$. Such behavior will always become subdominant for p_0 very close to p_c. Thus, in a finite dimension we always expect a crossover from an anomalous fluctuation regime to a critical regime, such as that displayed by the symmetric Cayley tree model. We thus conclude that in large dimensions the associated processes become objects with a fractal dimension approaching $D_\infty=7\pm1$. Since in six dimensions D, as well as other critical exponents, is close to its infinite-dimensional value, we are led to conjecture that there exists a finite critical dimension beyond which infinite-dimensional critical behavior is achieved.

IV. CONCLUSIONS

The analysis presented in this paper allows us to draw conclusions about the processes belonging to the quenched KPZ universality class and processes of percolation of directed surfaces. We believe that studying the behavior in different dimensionalities has provided many interesting insights. The main conclusions and ideas presented in this paper are as follows.

There exist two different roughness exponents characterizing the invading front: the exponent χ_c of the AP roughness and the overall interface roughness χ_q. The overall roughness is created by random deposition of objects with a roughness χ_c. We are in fact able to predict the exponent χ_q for $d>4$ using this understanding. A basic conclusion of this is that different driving methods of this model will display different statistical properties.

Another basic idea is that the dynamic process of avalanche formation below six dimensions is a percolative process. There are two main conclusions from this claim. First, the roughness exponent χ_c goes down with increasing dimension and the dimension where $\chi_c=0$ is $d_c=6$. Second, the dynamic exponents of the process should be the same as in percolation [12].

From our analysis on the Cayley tree we find that in high dimensions strong fluctuations appear as a result of exponentially rare events, as well as a result of critical behavior. These fluctuations are observed by the presence of wide tails in the AP size distribution and by power-law tails in the cluster distribution on the Cayley tree. As a result, one can observe two types of scenarios for the depinning transition on the Cayley tree. In addition to a critical transition, it is also possible to have a fluctuation dominated transition, characterized by a correlation length that does not diverge. Indeed, this behavior is observed both in a directed model we solve and in a model analyzed in [11]. On the other hand, on a simple symmetric bounded slope model the usual critical transition is realized. We conclude that the latter model is the appropriate high-dimensional limit.

The examination of the values of scaling exponents obtained numerically (see Table I) indicates that there is a finite absolute critical dimension in which all the exponents reach their infinite-dimensional limit.

In summary, we find that there are at least three qualitative transitions in the basic structure of an invading interface when dimensionality is increased. This reflects the wealth of processes occurring during such an invasion process.

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[1] P. Meakin, Phys. Rep. **235**, 189 (1993); T. Halpin-Healey and Y.-C. Zhang, *ibid.* **254**, 215 (1995); A.-L. Barabasi and H. E. Stanley, *Fractal Concepts in Surface Growth* (Cambridge University Press, Cambridge, 1995).

[2] M. A. Rubio, C. A. Edwards, A. Dougherty, and J. P. Gollub, Phys. Rev. Lett. **63**, 1685 (1990); V. K. Horváth, F. Family, and T. Vicsek, J. Phys. A **24**, L25 (1991).

[3] S. V. Buldyrev, A. L. Barabasi, F. Caserta, S. Havlin, H. E. Stanley, and T. Vicsek, Phys. Rev. A **45**, R8313 (1992).

[4] D. A. Kessler, H. Levine, and Y. Tu, Phys. Rev. A **43**, 4551 (1991); G. Parisi, Europhys. Lett. **17**, 673 (1992) T. Nattermann, S. Stepanow, L.-H. Tang, and H. Leschhorn, J. Phys. (France) II **2**, 1483 (1992); O. Narayan and D. S. Fisher Phys. Rev. B **48**, 7030 (1993); Z. Csahók, K. Honda, and T. Vicsek J. Phys. A **26**, L171 (1993); N. Martys, M. Cieplak, and M. O. Robbins, Phys. Rev. Lett. **66**, 1058 (1991); C. S. Nolle, B. Koiller, N. Martys, and M. O. Robbins, *ibid.* **71**, 2074 (1993).

[5] L.-H. Tang and H. Leschhorn, Phys. Rev. A **45**, R8309 (1992).

[6] M. Kardar, G. Parisi, and Y. C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).

[7] K. Sneppen, Phys. Rev. Lett. **69**, 3539 (1992).

[8] Z. Olami, I. Procaccia, and R. Zeitak, Phys. Rev. E **52**, 3402 (1995).

[9] Z. Olami, I. Procaccia, and R. Zeitak, Phys. Rev. E **49**, 1232 (1994).

[10] O. Gat and Z. Olami, Europhys. Lett. **36**, 49 (1996)

[11] S. V. Buldyrev, S. Havlin, J. Kertesz, R. Sadr-Lahijany, A. Shechter, and H. E. Stanley, Phys. Rev. E **52**, 373 (1995).

[12] S. Havlin, L. A. N. Amaral, S. V. Buldyrev, T. Harrington, and H. E. Stanley, Phys. Rev. Lett. **74**, 4205 (1995).

[13] T. Ala-Nissila, T. Hjelt, J. M. Kosterlitz, and O. Venäläin, J. Stat. Phys. **72**, 207 (1993).

[14] A. Bunde and S. Havlin, in *Fractals and Disordered Systems*, edited by A. Bunde and S. Havlin (Springer, Berlin, 1991).

[15] T. E. Harris, *The Theory of Branching Processes* (Springer, Berlin, 1963).

[16] W. Feller, *An Introduction to Probability Theory and its Applications* (Wiley, New York, 1967).